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On Relativistic Collisional Invariants

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Two simple proofs of the result that a relativistic summational invariant ψ is a linear combination of the momentum four-vector p^α are given by assuming that ψ is a continuous and differentiable function of class C^2. The results can be extended to the case when ψ is just assumed to be a generalized function.

KEY WORDS: Kinetic theory; relativity; collision invariants.

1. INTRODUCTION

The concept of collisional invariants plays an important role in the study of the non-relativistic and relativistic Boltzmann equations since it is related to the equilibrium distribution function.

For both cases the starting point is that the collision term of the Boltzmann equation must vanish in equilibrium, because of the H-theorem. This implies that the following condition must hold:

$$f(f(\mathbf{x}, \mathbf{p}, t)\, f(\mathbf{x}, \mathbf{p}_*, t) = f(\mathbf{x}, \mathbf{p}', t)\, f(\mathbf{x}, \mathbf{p}'_*, t) \tag{1}$$

where $f(\mathbf{x}, \mathbf{p}, t)$ and $f(\mathbf{x}, \mathbf{p}_*, t)$ represent distribution functions of two particles in a binary collision, that depend on the position $\mathbf{x}$, time t and precollisional momenta $\mathbf{p}$ and $\mathbf{p}_*$. Further $\mathbf{p}'$ and $\mathbf{p}'_*$ denote pre-collisional momenta which will be transformed into $\mathbf{p}$ and $\mathbf{p}_*$ by a collision.

The conservation laws of momentum and energy:

$$\mathbf{p} + \mathbf{p}_* = \mathbf{p}' + \mathbf{p}'_*, \qquad |\mathbf{p}|^2 + |\mathbf{p}_*|^2 = |\mathbf{p}'|^2 + |\mathbf{p}'_*|^2 \tag{2}$$

are satisfied in the classical case.

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For relativistic gases the energy-momentum conservation law is written as

$$p^\alpha + p^\alpha_* = p'^\alpha + p'^\alpha_* \tag{3}$$

where p^α, ($\alpha = 0, 1, 2, 3$) represent the components of the momentum four-vector of a particle in a Minkowski space of metric tensor $\eta^{\alpha\beta}$ with signature $(1, -1, -1, -1)$. Due to the constraint $p^\alpha p_\alpha = m^2c^2$ we have that $p^0 = \sqrt{|\mathbf{p}|^2 + m^2c^2}$.

Equation (1) can be written as

$$\psi(\mathbf{p}) + \psi(\mathbf{p}_*) = \psi(\mathbf{p}') + \psi(\mathbf{p}'_*) \tag{4}$$

where $\psi(p) = \ln f(\mathbf{x}, \mathbf{p}, t)$ is a so-called summational invariant.

The solution of (4) for non-relativistic monatomic gases is given by:

$$\psi(\mathbf{p}) = A + \mathbf{B} \cdot \mathbf{p} + C\,|\mathbf{p}|^2 \tag{5}$$

A and C are arbitrary scalars and $\mathbf{B}$ an arbitrary vector that do not depend on $\mathbf{p}$.

In the relativistic case the solution of (4) is

$$\psi(\mathbf{p}) = A + B^\alpha p_\alpha \tag{6}$$

where A is an arbitrary scalar and B^α an arbitrary four-vector that do not depend on p^α.

The first proof of (5) was given by Boltzmann[1,2] for twice differentiable functions and later several authors proved the same result under less stringent assumptions. Among others we cite the works of Gronwall,[3,4] Grad,[5] Carleman,[6] Hurley,[7] Arkeryd,[8] Truesdell and Muncaster,[9] Cercignani,[10] and Arkeryd and Cercignani.[11] Recently Wennberg[12] proved (5) under the assumption that ψ is a generalized function, or distribution.

For the relativistic case the proof of (6) was given by Chernikov,[13] Bichteler,[14] Boyer,[15] Marle,[16] Ehlers[17] and Dijkstra.[18]

The aim of this paper is to give two simple and slightly different proofs that (6) holds in the relativistic case. Our proofs have, with respect to those quoted above, the advantage of simplicity. The proof by Boyer[15] uses a 3-dimensional hyperbolic (or Lobachevski) space H_3 in order to reinterpret the ideas of relativistic kinematics, and proved the theorem as an application of this reinterpretation. Bichteler[14] and Ehlers[17] make use of Cartan's form calculus on manifolds of high dimension. Marle[16] and Chernikov[13] offer interesting but somewhat lengthy proofs in the case when the collision invariants are assumed to be continuous functions. Dijkstra[18] gave a somewhat technical proof in the case of measurable functions. Here we give two proofs which use essentially the argument used by Boltzmann in the classical case. This requires the seemingly stronger assumption that the collision invariants are twice differentiable, but thanks to the idea of Wennberg,[12] this is not a strong restriction, because one can assume that the collision invariants are distributions and the derivatives are taken in the sense of distributions. The advantage of this proof is that the property of being a collision invariant is shown to be a local one; the cross section might vanish for sets of nonzero measure (provided the complement is not of zero measure), but the set of collision invariants would remain the same.

2. STATEMENTS AND PROOFS

The content of this section is mainly devoted to the statement and proof of the following theorem:

Theorem 1. A continuous and differentiable function of class C^2 $\psi(p^\alpha)$ is a summational invariant if and only if it is given by (6), where A is an arbitrary scalar and B^α an arbitrary four-vector that do not depend on p^α.

As stated in the introduction, the theorem will be proved in two different ways. As a corollary, using Wennberg's argument,[12] we also prove

Theorem 2. A generalized function, or distribution, $\psi(p^\alpha)$ is a summational invariant if and only if it is given by (8), where A is an arbitrary scalar and B^α an arbitrary four-vector that do not depend on p^α.

First Proof of Theorem 1. Assume that $\psi(p^\alpha)$ is given by (6). Due to the conservation law of momentum four-vector (3) the relationship (6) satisfies (4) identically.

Next suppose that (6) holds. The conservation law of the momentum four-vector implies that there exists a function Ψ such that

$$\psi(\mathbf{p}) + \psi(\mathbf{p}_*) = \Psi(\pi, u) = \Psi(\pi', u') = \psi(\mathbf{p}') + \psi(\mathbf{p}'_*) \tag{7}$$

where

$$\pi = \mathbf{p} + \mathbf{p}_*, \qquad u = p_0 + p_{*0} = \sqrt{|\mathbf{p}|^2 + m^2c^2} + \sqrt{|\mathbf{p}_*|^2 + m^2c^2} \tag{8}$$

In fact the first and last expression in (7) must be equal for all the four-momentum vectors for which (4) holds and hence they must be both functions of the quantities which are invariant when we pass from the unprimed to the primed variables according to the definition of this transformation.

In the following we shall need to know the derivatives of p_0 with respect to $\mathbf{p}$, which are given by

$$\frac{\partial p_0}{\partial p^i} = -\frac{p_i}{p_0}, \qquad \frac{\partial^2 p_0}{\partial p_i \, \partial p_j} = -\left(\frac{p_i p_j}{p_0^3} + \frac{\eta_{ij}}{p_0}\right) \tag{9}$$

First we differentiate (7) with respect to $\mathbf{p}$ and get

$$\frac{\partial \psi}{\partial p^i} = \frac{\partial \Psi}{\partial \pi^i} - \frac{\partial \Psi}{\partial u} \frac{p_i}{p_0} \tag{10}$$

By applying the same procedure we differentiate (7) with respect to $\mathbf{p}_*$ and write the difference

$$\frac{\partial \psi}{\partial p^i} - \frac{\partial \psi}{\partial p^i_*} = -\frac{\partial \Psi}{\partial u}\left(\frac{p_i}{p_0} - \frac{p_{*i}}{p_{*0}}\right) \tag{11}$$

which implies that

$$\left(\frac{\partial \psi}{\partial p^i} - \frac{\partial \psi}{\partial p^i_*}\right)\left(\frac{p_j}{p_0} - \frac{p_{*j}}{p_{*0}}\right) = \left(\frac{\partial \psi}{\partial p^j} - \frac{\partial \psi}{\partial p^j_*}\right)\left(\frac{p_i}{p_0} - \frac{p_{*i}}{p_{*0}}\right) \tag{12}$$

Next the differentiation of the above equation with respect to p^k, yields

$$\begin{aligned} &\frac{\partial^2 \psi}{\partial p^i \, \partial p^k}\left(\frac{p_j}{p_0} - \frac{p_{*j}}{p_{*0}}\right) + \left(\frac{\partial \psi}{\partial p^i} - \frac{\partial \psi}{\partial p^i_*}\right)\left(\frac{p_j p_k}{p_0^3} + \frac{\eta_{jk}}{p_0}\right) \\ &\quad = \frac{\partial^2 \psi}{\partial p^j \, \partial p^k}\left(\frac{p_i}{p_0} - \frac{p_{*i}}{p_{*0}}\right) + \left(\frac{\partial \psi}{\partial p^j} - \frac{\partial \psi}{\partial p^j_*}\right)\left(\frac{p_i p_k}{p_0^3} + \frac{\eta_{ik}}{p_0}\right) \end{aligned} \tag{13}$$

Further the differentiation of (13) with respect to p^l_* reads

$$\begin{aligned} &\frac{\partial^2 \psi}{\partial p^i \, \partial p^k}\left(\frac{p_{*j} p_{*l}}{p_{*0}^3} + \frac{\eta_{jl}}{p_{*0}}\right) + \frac{\partial^2 \psi}{\partial p^i_* \, \partial p^l_*}\left(\frac{p_j p_k}{p_0^3} + \frac{\eta_{jk}}{p_0}\right) \\ &\quad = \frac{\partial^2 \psi}{\partial p^j \, \partial p^k}\left(\frac{p_{*i} p_{*l}}{p_{*0}^3} + \frac{\eta_{il}}{p_{*0}}\right) + \frac{\partial^2 \psi}{\partial p^j_* \, \partial p^l_*}\left(\frac{p_i p_k}{p_0^3} + \frac{\eta_{ik}}{p_0}\right) \end{aligned} \tag{14}$$

Equation (14) has the following form

$$C_{ik}(\mathbf{p})\, D_{jl}(\mathbf{p}_*) + C_{il}(\mathbf{p}_*)\, D_{jk}(\mathbf{p}) = C_{jk}(\mathbf{p})\, D_{il}(\mathbf{p}_*) + C_{jl}(\mathbf{p}_*)\, D_{ik}(\mathbf{p}) \tag{15}$$

Equation (15) is a tensorial equation of fourth order, the fourth order tensors being products of second order tensors that depend on different variables $\mathbf{p}$ and $\mathbf{p}_*$. It will be satisfied only if

$$C_{ik}(\mathbf{p}) = -B^0 D_{ik}(\mathbf{p}), \quad \text{or} \quad \frac{\partial^2 \psi}{\partial p^i \, \partial p^k} = -B^0 \left(\frac{p_i p_k}{p_0^3} + \frac{\eta_{ik}}{p_0} \right) = B^0 \frac{\partial^2 p_0}{\partial p^i \, \partial p^k} \tag{16}$$

where B^0 is a scalar that does not depend on p^α. Now the integration of $(16)_2$ leads to

$$\psi(\mathbf{p}) = A + B^i p_i + B^0 p_0, \qquad \text{or} \quad \psi(\mathbf{p}) = A + B^\alpha p_\alpha \tag{17}$$

Here A is a scalar and B^i are the components of a three-dimensional vector that do not depend on p^α. Hence we have proved Theorem 1.

The proof given above is based on that given by Boltzmann,[1, 2] as rewritten by Cercignani,[10] for the non-relativistic case.

Second Proof of Theorem 1. The previous proof is made a bit lengthy by the necessity of taking the constraint $p^\alpha p_\alpha = m^2c^2$ into account. This is necessary, if we want to stick to physics. If we want to give a mathematical proof only, we can assume that the speed of light c is also an independent variable and the constraint disappears because p_0 can be taken as an independent variable in place of c. In other words, we embed our problem into a larger one, by letting a parameter vary. This is, however, not enough, because we need p_{*0} to be independent as well. Thus we relax the constraint further and replace it by a fifth conservation law:

$$p^\alpha p_\alpha + p_*^\alpha p_{*\alpha} = p'^\alpha p'_\alpha + p'^\alpha_* p'_{*\alpha} \tag{18}$$

which is trivially satisfied in the original problem.

Then we can repeat the nonrelativistic proof, since (3) and (18) form exactly the same system (except for the trivial changes due to the pseudo-Euclidean metric) as in the classical case.[10] Of course, we are now working in four dimensions. The proof now yields the following result:

$$\psi(\mathbf{p}) = A + B^\alpha p_\alpha + C p^\alpha p_\alpha \tag{19}$$

where A and C are scalar constants, B^α the components of a four-dimensional constant vector. Now we introduce the constraint $p^\alpha p_\alpha = m^2c^2$ and we obtain (6) except for a trivial change (the constant A is replaced by another constant $A' = A + Cm^2c^2$).

We come now to the second theorem stated above.

Proof of Theorem 2. Since our problem is linear, we can interpret all our manipulations even if we assume that ψ is just a generalized function, or distribution, for which derivatives of any order exist, as first pointed out by Wennberg.[12] The result remains unchanged and Theorem 2 holds.

Remark. The result of Theorem 2 can be useful in several situations. In particular, ψ can be an ordinary function, locally integrable, but without any smoothness property.

3. CONCLUDING REMARKS

The proofs given in this paper indicate how to generalize the simplest proof on the formula for the collision invariants to the case of a relativistic gas. We have given two proofs: the second one is somewhat artificial from a physical viewpoint, but has the advantage of showing that any proof given in the classical case can be transferred to the relativistic one, by just embedding the problem in a wider one and introducing a fifth conservation law in place of the constraint on the components of the four-momentum vector. Finally, the proof of Theorem 2, based on a result first provided by Wennberg in the classical case, shows that the smoothness assumptions can be considerably relaxed.

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